Area Preserving Maps on S^2: A Lower Bound on the C^0-norm using Symplectic Spectral Invariants

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Abstract

We use the Hofer norm to show that all Hamiltonian diffeomorphisms with compact support in $\mathbb{R}^{2n}$ that displace an open set with a nonzero Hofer-Zehnder capacity move a point farther than a capacity-dependent constant. In $\mathbb{R}^2$, this result is extended to all compactly supported symplectomorphisms. Next, using the spectral norm, we show the result holds for Hamiltonian diffeomorphisms on closed surfaces. We then show that all area-preserving homeomorphisms of S^2 and $\mathbb{RP}^2$ that displace the closure of an open path connected set of fixed area move a point farther than an area-dependent constant.

1 Introduction

The problem which motivates this paper was originally posed as below:

Question 1. *Given a fixed area $A > 0$, is there a constant $\delta(A) > 0$, such that for any homeomorphism $f : S^2 \to S^2$ which preserves area and displaces a subset $E \subset S^2$ with area A , there is an $x \in S^2$ with $|f(x) - x| \geq \delta(A)$? What conditions are needed on E?*

We say that a function displaces a set E if $f(E) \cap E = \emptyset$ throughout. Despite its rather concrete and explicit nature, the problem seems to be very difficult to solve using elementary techniques. Due to an idea originally of Dmitri Burago, Sergei Ivanov, and Leonid Polterovich, we approached the problem using recently discovered methods from symplectic topology. In particular, we use the Hofer norm $\|\psi\|$ to demonstrate a similar property for any Hamiltonian diffeomorphism of $\mathbb{R}^{2n}$ and use the spectral norm $\gamma(\phi)$ to obtain the same result for any Hamiltonian diffeomorphism of a closed surface. By applying additional properties of S^2, we extend the result to any homeomorphism that displaces the closure of an open, path-connected set to obtain an affirmative answer to the question posed above. Finally, we show by lifting to the universal cover that the same result holds for $\mathbb{RP}^2$.

Authors' Note: Although we were unaware of it during the production of this paper, it has been brought recently to our attention that Sobhan Seyfaddini posted a paper with similar results [1]. Though our paper is entirely independent of his, we acknowledge that his previous work provided the key result of C^0-continuity of the spectral norm. We believe that the different emphasis and unique extensions in our paper make it of independent interest.

2 Results

First, we prove a similar result that can be obtained using the Hofer norm. Let (M, ω) be a symplectic manifold. Any compactly supported Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$ generates a time-dependent vector field X_H defined by $i_{X_H}\omega = dH_t$ where $H_t(x) = H(t,x)$ and induces a Hamiltonian flow $\phi_H^t : M \to M$. The Hofer norm on the space of compactly supported Hamiltonians is

$$\|H\| = \int_0^1 \max_x \; H(t,x) - \min_x \; H(t,x) \; dt.$$

A Hamiltonian diffeomorphism is the time-1 map of a Hamiltonian flow. Denote the group of compactly supported Hamiltonian diffeomorphisms on (M,ω) by $Ham_c(M,\omega)$. The *Hofer norm* of $\psi \in Ham_c(M,\omega)$ is defined as

$$\|\psi\| = \inf_H \left\{ \|H\| : \psi = \phi_H^1 \right\}$$

and although first developed for $(\mathbb{R}^{2n}, \omega_0)$, is a norm for arbitrary symplectic manifolds. In [2], Hofer showed that there exists a constant $C > 0$ such that for $\psi \in Ham_c(\mathbb{R}^{2n}, \omega_0)$ and D = diameter supp(ψ):

$$\|\psi\| \leq CD\|\psi\|_{C^0} = CD \sup_x \|\psi(x) - x\|$$

establishing the C^0-continuity of the Hofer norm in $(\mathbb{R}^{2n}, \omega_0)$. He also introduces the *displacement energy* of an open set $U \subset \mathbb{R}^{2n}$:

$$e(U) = \inf_{\psi \in Ham_c(\mathbb{R}^{2n},\omega_0)} \{\|\psi\| : \psi(U) \cap U = \emptyset\}$$

where $e(U) = \infty$ if U cannot be displaced such as when volume$(U) > \frac{1}{2}$ volume(M). He also proves the energy-capacity inequality:

$$c_{HZ}(U) \leq e(U)$$

where the Hofer-Zehnder capacity, c_{HZ}, is defined by

$$c_{HZ}(M,\omega) = \sup\{\|H\| : H \text{ is admissible}\}$$

where a Hamiltonian function is admissible if its flow has no nonconstant periodic orbits with period ≤ 1. We now use these results to obtain a result in the spirit of Question 1:

Theorem 2.1. *There exists $\delta(A,D) > 0$ such that for any $\psi \in Ham_c(\mathbb{R}^{2n}, \omega_0)$ with D = diameter supp(ψ) that displaces an open set $U \subset \mathbb{R}^{2n}$ with $c_{HZ}(U) = A > 0$, we have $\|x - \psi(x)\| \geq \delta(A,D)$ for some $x \in \mathbb{R}^{2n}$.*

Proof. By the C^0 relation to the Hofer norm and the energy-capacity inequality we have:

$$0 < A = c_{HZ}(U) \leq \|\psi\| \leq CD\|\psi\|_{C^0}$$

which implies the result. □

We would like to extend this result to the group of compactly supported symplectomorphisms, that is, $\{\psi \in \text{Diff}(M) : \psi^*\omega = \omega \text{ and } \psi(x) = x \text{ outside a compact set}\}$. We denote this group $Symp_c(M, \omega)$. When the first (de Rham) cohomology group of M is trivial, the connected component of the identity, $Symp_{c,0}(M, \omega)$, is equal to $Ham_c(M, \omega)$. Unfortunately, little is known about $Symp_c(\mathbb{R}^{2n}, \omega_o)$ for $n > 2$. For $n = 2$, M. Gromov showed in [3] that $Symp_c(\mathbb{R}^4, \omega_o)$ is contractible which means Theorem 2.1 can be extended to the whole group. For $n = 1$, we have the following:

Corollary 2.2. *There exists $\delta(A, D) > 0$ such that for any $\psi \in Symp_c(\mathbb{R}^2, \omega_0)$ with $D =$ diameter supp(ψ) that displaces an open path connected set $U \subset \mathbb{R}^2$ with Area(U) $= A > 0$, we have $\|x - \psi(x)\| \geq \delta(A, D)$ for some $x \in \mathbb{R}^2$.*

Proof. First, we note that U is bounded because ψ has compact support. For an open, bounded, path connected subset $U \subset \mathbb{R}^2$, Area$(U) = c_{HZ}(U)$. (See Chapter 3 of [4]).

Since the support of ψ is in a closed ball, B^2, of diameter D, it can be regarded as an element of $Symp_0(B^2, \omega_o)$. Moreover, ψ fixes the boundary of B^2 since $\{x \in \mathbb{R}^2 : \psi(x) = x\}$ is closed.

We first show $Symp_0(B^2, \omega_0) = Ham(B^2, \omega_0)$. We have that ψ corresponds to a unique time-dependent vector field X_t with $\frac{d}{dt}\psi_t = X_t \circ \psi_t$ for $t \in [0, 1]$, $\phi = \phi_1$, and $\phi_0 = id$. This vector field is canonically paired with a one-form $\sigma = i_{X_t}\omega$ defined by $i_{X_t}\omega(Y) = \omega(Y, X_t)$. Since $H^1(B^2, \mathbb{R}) = 0$, σ is exact: $\sigma = dH_t$ for some function $H_t : B^2 \to \mathbb{R}$.

Now, $Symp(B^2, \omega_o)$ is a retract of the set of orientation diffeomorphisms which preserve the boundary by Moser's Theorem. S. Smale showed in [5] that the second group is contractible, which implies $Symp(B^2, \omega_o)$ is path connected.

Therefore, $\psi \in Ham_c(\mathbb{R}^2, \omega_0)$ and we get the result by applying Theorem 2.1. □

In this case, our constant depends both on the area and the diameter of the support of the function. It is clear that in the case of a compact manifold, the diameter is removed. Unfortunately, the Hofer norm is not C^0-continuous on closed manifolds [6] so this method cannot be used to answer Question 1, but we follow a very similar approach below.

We now turn to the case of a closed symplectic manifold (M, ω) and introduce tools to handle this case. In [7], Y. G. Oh defines the *spectral norm*, $\gamma : Ham(M, \omega) \to \mathbb{R}_+$ using spectral invariants obtained from the Floer homology theory. It satisfies:

1. $\phi = id$ iff $\gamma(\phi) = 0$ for all $\phi \in Ham(M, \omega)$
2. $\gamma(\eta\phi\eta^{-1}) = \gamma(\phi)$ for all $\phi \in Ham(M, \omega)$ and all $\eta \in Symp(M, \omega)$
3. $\gamma(\phi\psi) \leq \gamma(\phi) + \gamma(\phi)$ for all $\phi, \psi \in Ham(M, \omega)$
4. $\gamma(\phi^{-1}) = \gamma(\phi)$ for all $\phi \in Ham(M, \omega)$
5. $\gamma(\phi) \leq \|\phi\|$ for all $\phi \in Ham(M, \omega)$

The most difficult property to prove is the non-degeneracy of γ and is critical to the results obtained here. S. Seyfaddini proved in [6] that the spectral norm is C^0-continuous on the space of Hamiltonian diffeomorphisms on two-dimensional manifolds by showing that there exists $C, d > 0$ such that for $\|\phi\|_{C^0} \leq d$ we have

$$\gamma(\phi) \leq C(\|\phi\|_{C^0})^{2^{-2g-1}}$$

where g is the genus of the surface and $\|\phi\|_{C^0} = \sup_x \|\phi(x) - x\|$ with a Riemannian metric $\|\cdot\|$ on M. It has not been established if a similar bound holds in higher dimensions. Therefore, we restrict all of our results to the two-dimensional case. Providing a counter-example to Theorem 2.3 in higher dimensions would show that no such bound exists. However, Theorem 2.1 shows that a similar result holds in $\mathbb{R}^{2n}$ so it seems possible that Theorem 2.3 could hold in higher dimensions.

Analogously to the case of the Hofer norm, we can define the *spectral displacement energy* following Oh in [7] for an open set $U \subset M$ by:

$$e_\gamma(U) = \inf_{\phi \in Ham(M,\omega)} \{\gamma(\phi) : \phi(U) \cap U = \emptyset\}$$

The other key element is the spectral energy-capacity relation recently shown by M. Usher in [8] between $e_\gamma(U)$ and the inner Hofer-Zehnder capacity $c^o_{HZ}(U)$

$$e_\gamma(U) \geq c^o_{HZ}(U) \geq c_{HZ}(U)$$

where the inner Hofer-Zehnder capacity is defined by

$$c^o_{HZ}(M, \omega) = \sup\{\|H\| : H \text{ is inner admissible}\}$$

where a Hamiltonian function is inner admissible if its flow has no nonconstant contractible periodic orbits with period ≤ 1. We can now prove the analog of Theorem 2.1 for 2-dimensional closed manifolds.

Theorem 2.3. *Let (M, ω) be a closed 2-dimensional symplectic manifold. There exists $\delta(A) > 0$ such that for any $\phi \in Ham(M, \omega)$ that displaces an open set $U \subset M$ with $c_{HZ}(U) = A > 0$, we have $\|x - \phi(x)\| \geq \delta(A)$ for some $x \in M$.*

Proof. Let C, d be as in the C^0 relation to the spectral norm above. Assume there is such a ϕ with $\|\phi\|_{C^0} \leq d$. Otherwise, let $\delta(A) = d$. Then, we apply the spectral energy-capacity inequality to get

$$0 < A = c_{HZ}(U) \leq \gamma(\phi) \leq C(\|\phi\|_{C^0})^{2^{-2g-1}}$$

which implies the result. □

In fact, the spectral norm exists on a wider class of manifolds. One particular case where the spectral norm is continuous is on $Ham_c(T^*_r N, \omega_o)$ where $T^*_r N$ is the cotangent ball bundle of radius r and ω_o is the canonical two-form on T^*N. The continuity of γ was shown by S. Seyfaddini in [9] (Theorem 5, $\lambda = 0$). From this and the energy-capacity inequality, we get the following theorem (a generalization of Theorem 2.1) for which the proof is omitted since it is very similar to the proofs of Theorems 2.1 and 2.3.

Theorem 2.4. *Let (T^*N, ω_o) be a cotangent bundle with the canonical two-form. There exists $\delta(A, D) > 0$ such that for any $\phi \in Ham_c(T^*N, \omega_o)$ with $D =$ diameter supp(ϕ) that displaces an open set $U \subset T^*N$ with $c_{HZ}(U) = A > 0$, we have $\|x - \phi(x)\| \geq \delta(A, D)$ for some $x \in T^*N$.*

It is important to note that Theorem 2.3 only holds for Hamiltonian diffeomorphisms. In the case of $(\mathbb{T}^2, \omega_0)$, one can consider the function that rotates the torus about its central axis, which is not Hamiltonian since it is the flow of a vector field corresponding to a nonzero element of $H^1(M, \omega)$. Then, fix the area A and create a thin set that wraps around the torus with this area. This set can be made arbitrarily thin and a small rotation will displace this set so the result does not hold for all symplectomorphisms of $\mathbb{T}^2$.

As in the case of $\mathbb{R}^2$, we can extend the theorem for S^2 by showing that all symplectomorphisms are Hamiltonian. In addition, we can prove the result for area-preserving homeomorphisms by approximating them with symplectomorphisms in order to answer Question 1.

Theorem 2.5. *Let (S^2, ω_o) be the sphere with the canonical two-form. There exists $\delta(A) > 0$ such that for any area-preserving homeomorphism $f : S^2 \to S^2$ that displaces the closure of an open path connected set $U \subset S^2$ with Area$(U) = A > 0$, we have $\|x - f(x)\| \geq \delta(A)$ for some $x \in S^2$.*

Proof. First, note that if f reverses orientation, it maps at least one point to the antipodal point (otherwise a homotopy to the identity can be constructed along the shortest paths). As this is the maximal displacement of a point on the sphere, it must be the case that $\|f\|_{C^0} \geq \delta(A)$, where $\delta(A)$ is the lower bound we will establish for orientation preserving homeomorphisms. For the orientation preserving case, we need the symplectic theory.

By the same argument presented in the proof of Corollary 2.2, $Ham(S^2, \omega_o) = Symp_0(S^2, \omega_0)$. Now, we observe that $Symp_0(S^2, \omega_0) = Symp(S^2, \omega_0)$. By Moser's theorem, $Symp(S^2, \omega_0)$ is a retract of $Diff^+(S^2)$. S. Smale showed in [5] that the second group is homotopic to the path connected group $SO(3)$. Therefore, every $\phi \in Symp(S^2, \omega_0)$ is Hamiltonian. Also, $c_{HZ}(U) =$Area(U) for open path connected sets $U \subset S^2$ (The same proof given in [4] for $\mathbb{R}^2$ can easily be extended to S^2). Thus, Theorem 2.3 can be applied to obtain $\delta(A)$ for all symplectomorphisms.

Next, we show that this property extends to orientation and area preserving homeomorphisms. Y. G. Oh showed in [10] that on a closed surface, an orientation and area preserving homeomorphism can be C^0 approximated by an orientation and area preserving diffeomorphism (symplectomorphism). This says that in the 2-dimensional case, all orientation and area preserving homeomorphisms are symplectic homeomorphisms. Thus, if U is an open set with Area$(U) = A > 0$ and f is a symplectic homeomorphism such that $f(\overline{U}) \cap \overline{U} = \emptyset$, then there is a symplectomorphism ϕ such that $\sup_{x \in M} \|f(x) - \phi(x)\| < \varepsilon$ for any ε. Since $\overline{U}$ is compact, we can choose ϵ small enough such that $\phi(\overline{U}) \cap \overline{U} = \emptyset$. Then, the previous result shows that there is an x such that $\|\phi(x) - x\| \geq \delta(A)$, so $\|f(x) - x\| \geq \delta(A) - \varepsilon$. Since ε is arbitrary, we have $\|f(x) - x\| \geq \delta(A)$. □

Corollary 2.6. *There exists $\delta(A) > 0$ such that for any area-preserving homeomorphism $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ that displaces the closure of an open path connected set $U \subset \mathbb{RP}^2$ with area A, we have $\|x - f(x)\| \geq \delta(A)$ for some $x \in \mathbb{RP}^2$.*

Proof. First we lift f to the unique orientation-preserving homeomorphism $g : S^2 \to S^2$.

We observe that g must be measure-preserving as well. Let $B \subset S^2$ be an open ball with area less than half of S^2 such that $g(B)$ also has area less than half of S^2. Let the standard measure on S^2 and on $\mathbb{RP}^2$ be denoted by μ, i.e., $\mu(A) = \text{Area}(A)$. Now, since f is measure preserving, $\mu(f \circ p(B)) = \mu(p(B))$. But since $f \circ p = p \circ g$, $\mu(p(B)) = \mu(p \circ g(B))$. But both B and $g(B)$ project to homeomorphic images on $\mathbb{RP}^2$, $\mu(p(B)) = \mu(B)$ and $\mu(p \circ g(B)) = \mu(g(B))$, so $\mu(g(B)) = \mu(B)$. Since g is locally measure-preserving, it is measure-preserving.

Next, g displaces the closure of a path connected open set with area greater than or equal to A: since f displaces $\overline{U}$, g displaces $\overline{p^{-1}(U)} = p^{-1}(\overline{U})$ (the equality follows from the fact that p is a local homeomorphism). To see this, take $x \in g(p^{-1}(\overline{U}))$. Then there is a $y \in p^{-1}(\overline{U})$ such that $g(y) = x$. Then $p(y) \in \overline{U}$ and $f \circ p(y) = p \circ g(y) = p(x) \in f(\overline{U})$. Since f displaces $\overline{U}$, $p(x)$ cannot be in $\overline{U}$, so $p^{-1}(\overline{U}) \cap g(p^{-1}(\overline{U})) = \emptyset$. By continuity, $p^{-1}(U)$ is open. Now, there are up to two path connected components of $p^{-1}(U)$ since any path can be lifted. If there are two path connected components, choose one since both will have area $= A$. If there is one path connected component, it must have area greater than or equal to A. Thus, g satisfies the conditions of Theorem 2.5 and there exists $\delta'(A) > 0$ and $x \in S^2$ such that $\|x - g(x)\| \geq \delta'(A)$.

Now, we show that this property descends to f. For convenience, normalize the metric so that the antipodal distance on S^2 is 1. First, if $\delta'(A) \leq \|x - g(x)\| < \frac{1}{2}$, let $\delta(A) = \delta'(A)$. Then $g(x)$ is outside the $\delta(A)$-ball at x and the $\delta(A)$-ball at the antipodal point of x, so $p(g(x)) = f(p(x))$ is outside the $\delta(A)$ ball centered at $p(x)$, giving the desired result. If $\frac{1}{2} \leq \|x - g(x)\|$, first let $\delta(A) = \min\{\delta'(A), \frac{1}{2} - \epsilon\}$ for any small $\epsilon > 0$. The result still holds: Since g is an orientation-preserving homeomorphism, it has a fixed point y by the Lefschetz theorem. Since the function from $S^2 \times S^2$ to $\mathbb{R}$ given by $\|x - g(x)\|$ is continuous and is zero at y, the intermediate value theorem implies that there is an x' such that $\|x' - g(x')\| \in [\delta(A), \frac{1}{2})$, so the previous argument applies. □

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